less than 0.005 that the combined resistance of the circuit would exceed 19 ohms?

4.3.36. The cylinders and pistons for a certain internal combustion engine are manufactured by a process that gives a normal distribution of cylinder diameters with a mean of 41.5 cm and a standard deviation of 0.4 cm. Similarly, the distribution of piston diameters is normal with a mean of 40.5 cm and a standard deviation of 0.3 cm. If the piston diameter is greater than the cylinder diameter, the former can be reworked until the two "fit." What proportion of cylinder-piston pairs will need to be reworked?

4.3.37. Use moment-generating functions to prove the two corollaries to Theorem 4.3.3.

4.3.38. Let $Y_1, Y_2, \ldots, Y_9$ be a random sample of size 9 from a normal distribution where $\mu = 2$ and $\sigma = 2$. Let $Y_1^*, Y_2^*, \ldots, Y_9^*$ be an independent random sample from a normal distribution having $\mu = 1$ and $\sigma = 1$. Find $P(\bar{Y} \geq \bar{Y}^*)$.

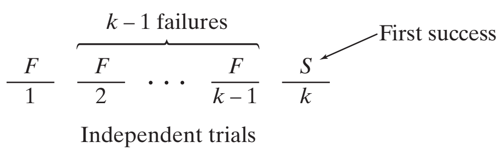
4.4 The Geometric Distribution

Consider a series of independent trials, each having one of two possible outcomes, success or failure. Let $p = P(\text{Trial ends in success})$. Define the random variable X to be the trial *at which the first success occurs*. Figure 4.4.1 suggests a formula for the pdf of X:

$$\begin{aligned} p_X(k) &= P(X = k) = P(\text{First success occurs on } k\text{th trial}) \\ &= P(\text{First } k-1 \text{ trials end in failure and } k\text{th trial ends in success}) \\ &= P(\text{First } k-1 \text{ trials end in failure}) \cdot P(k\text{th trial ends in success}) \\ &= (1-p)^{k-1} p, \quad k = 1, 2, \ldots \end{aligned} \tag{4.4.1}$$

We call the probability model in Equation 4.4.1 a *geometric distribution* (with parameter p).

Figure 4.4.1



Comment Even without its association with independent trials and Figure 4.4.1, the function $p_X(k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$ qualifies as a discrete pdf because (1) $p_X(k) \geq 0$ for all k and (2) $\sum_{\text{all } k} p_X(k) = 1$:

$$\begin{aligned} \sum_{k=1}^{\infty} (1-p)^{k-1} p &= p \sum_{j=0}^{\infty} (1-p)^j \\ &= p \cdot \left[\frac{1}{1-(1-p)} \right] \\ &= 1 \end{aligned}$$

Example 4.4.1 A pair of fair dice are tossed until a sum of 7 appears for the first time. What is the probability that more than four rolls will be required for that to happen?

Each throw of the dice here is an independent trial for which

$$p = P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$$

Let X denote the roll at which the first sum of 7 appears. Clearly, X has the structure of a geometric random variable, and

$$P(X > 4) = 1 - P(X \le 4) = 1 - \sum_{k=1}^{4} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$
$$= 1 - \frac{671}{1296}$$
$$= 0.48$$

Theorem 4.4.1 *Let X have a geometric distribution with $p_X(k) = (1-p)^{k-1}p, k = 1, 2, \ldots$. Then*

1. $M_X(t) = \frac{pe^t}{1-(1-p)e^t}$
2. $E(X) = \frac{1}{p}$
3. $\text{Var}(X) = \frac{1-p}{p^2}$

Proof See Examples 3.12.1 and 3.12.5 for derivations of $M_X(t)$ and $E(X)$. The formula for $\text{Var}(X)$ is left as an exercise. □

Example 4.4.2 A grocery store is sponsoring a sales promotion where the cashiers give away one of the letters A, E, L, S, U, or V for each purchase. If a customer collects all six (spelling *VALUES*), he or she gets \$10 worth of groceries free. What is the expected number of trips to the store a customer needs to make in order to get a complete set? Assume the different letters are given away randomly.

Let X_i denote the number of purchases necessary to get the ith different letter, $i = 1, 2, \ldots, 6$, and let X denote the number of purchases necessary to qualify for the \$10. Then $X = X_1 + X_2 + \cdots + X_6$ (see Figure 4.4.2). Clearly, X_1 equals 1 with probability 1, so $E(X_1) = 1$. Having received the first letter, the chances of getting a different one are $\frac{5}{6}$ for each subsequent trip to the store. Therefore,

$$f_{X_2}(k) = P(X_2 = k) = \left(\frac{1}{6}\right)^{k-1} \frac{5}{6}, \quad k = 1, 2, \ldots$$

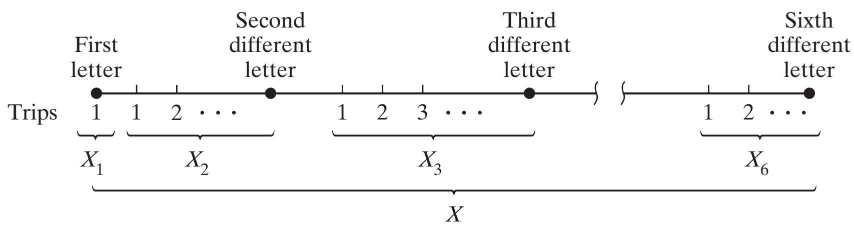


Figure 4.4.2

That is, X_2 is a geometric random variable with parameter $p = \frac{5}{6}$. By Theorem 4.4.1, $E(X_2) = \frac{6}{5}$. Similarly, the chances of getting a *third* different letter are $\frac{4}{6}$ (for each purchase), so

$$f_{X_3}(k) = P(X_3 = k) = \left(\frac{2}{6}\right)^{k-1} \left(\frac{4}{6}\right), \quad k = 1, 2, \ldots$$

and $E(X_3) = \frac{6}{4}$. Continuing in this fashion, we can find the remaining $E(X_i)$'s. It follows that a customer will have to make *14.7* trips to the store, on the average, to collect a complete set of six letters:

$$E(X) = \sum_{i=1}^{6} E(X_i)$$
$$= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$$
$$= 14.7$$

■

Questions

4.4.1. Because of her past convictions for mail fraud and forgery, Jody has a 30% chance each year of having her tax returns audited. What is the probability that she will escape detection for at least three years? Assume that she exaggerates, distorts, misrepresents, lies, and cheats every year.

4.4.2. A teenager is trying to get a driver's license. Write out the formula for the pdf $p_x(k)$, where the random variable X is the number of tries that he needs to pass the road test. Assume that his probability of passing the exam on any given attempt is 0.10. On the average, how many attempts is he likely to require before he gets his license?

4.4.3. Is the following set of data likely to have come from the geometric pdf $p_X(k) = \left(\frac{3}{4}\right)^{k-1} \cdot \left(\frac{1}{4}\right)$, $k = 1, 2, \ldots$? Explain.

2	8	1	2	2	5	1	2	8	3
5	4	2	4	7	2	2	8	4	7
2	6	2	3	5	1	3	3	2	5
4	2	2	3	6	3	6	4	9	3
3	7	5	1	3	4	3	4	6	2

4.4.4. Recently married, a young couple plans to continue having children until they have their first girl. Suppose the probability that a child is a girl is $\frac{1}{2}$, the outcome of each birth is an independent event, and the birth at which the first girl appears has a geometric distribution. What is the couple's expected family size? Is the geometric pdf a reasonable model here? Discuss.

4.4.5. Show that the cdf for a geometric random variable is given by $F_X(t) = P(X \le t) = 1 - (1-p)^{[t]}$, where $[t]$ denotes the greatest integer in t, $t \ge 0$.

4.4.6. Suppose three fair dice are tossed repeatedly. Let the random variable X denote the roll on which a sum of 4 appears for the first time. Use the expression for $F_x(t)$ given in Question 4.4.5 to evaluate $P(65 \le X \le 75)$.

4.4.7. Let Y be an exponential random variable, where $f_Y(y) = \lambda e^{-\lambda y}$, $0 \le y$. For any positive integer n, show that $P(n \le Y \le n+1) = e^{-\lambda n}(1 - e^{-\lambda})$. Note that if $p = 1 - e^{-\lambda}$, the "discrete" version of the exponential pdf is the geometric pdf.

4.4.8. Sometimes the geometric random variable is defined to be the number of trials, *X, preceding* the first success. Write down the corresponding pdf and derive the moment-generating function for X two ways—(1) by evaluating $E(e^{tX})$ directly and (2) by using Theorem 3.12.3.

4.4.9. Differentiate the moment-generating function for a geometric random variable and verify the expressions given for $E(X)$ and $\text{Var}(X)$ in Theorem 4.4.1.

4.4.10. Suppose that the random variables X_1 and X_2 have mgfs $M_{X_1}(t) = \frac{\frac{1}{2}e^t}{1-\left(1-\frac{1}{2}\right)e^t}$ and $M_{X_2}(t) = \frac{\frac{1}{4}e^t}{1-\left(1-\frac{1}{4}t\right)e^t}$, respectively. Let $X = X_1 + X_2$. Does X have a geometric distribution? Assume that X_1 and X_2 are independent.

4.4.11. The *factorial moment-generating function* for any random variable W is the expected value of t^w. Moreover $\frac{d^r}{dt^r}E(t^W)\mid_{t=1} = E[W(W-1)\cdots(W-r+1)]$. Find the factorial moment-generating function for a geometric random variable and use it to verify the expected value and variance formulas given in Theorem 4.4.1.

4.5 The Negative Binomial Distribution

The geometric distribution introduced in Section 4.4 can be generalized in a very straightforward fashion. Imagine waiting for the rth (instead of the first) success in a series of independent trials, where each trial has a probability of p of ending in success (see Figure 4.5.1).